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The transient heaving motion of floating cylinders

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SUMMARY

The transient heave response of a freely floating cylinder with given initial conditions is obtained by a simultaneous time-domain solution of the fluid-motion and rigid-body dynamics problems. Volterra's method is used to derive the integral equation associated with the fluid motion. It is shown that the unit initial-velocity response is simply the time-derivative of the unit initial-displacement response multiplied by one half of the infinite-fluid virtual mass of the cylinder. Numerical evaluation of integrals related to the unsteady water-wave Green function is facilitated by expressing them in terms of the complex error function. Results for the transient motion of semi-circular, triangular, and rectangular cylinders are presented and discussed. Experimental measurements for the case of a semi-immersed circular cylinder agree excellently with the theoretical calculations.

1. Introduction

The hydrodynamics of bodies oscillating steadily in a free surface is characterized by frequency-dependent force coefficients. In the time domain, such dependence is equivalent to the presence of memory effects. The problem of determining the transient response of a freely floating body has the additional interesting feature that the instantaneous force is coupled with the unknown body motion.

Interests in the solution of such unsteady problems arose in the mid-60s, during which experimentalists investigated the possibility of determining the frequency spectrum of the force coefficients by using the results of single transient-response experiments. The theoretical basis for such an approach was first outlined by Cummins [5], who introduced a step-response function, based on which the time-dependent force and moment on a body can be evaluated. Kotik and Lurye [12] considered the generalization of such an approach to include forward-speed effects. The actual computation of the unit-step response function was nontrivial and not considered in either work. Wehausen [17] provided a broader formulation and derived the associated integral equation for the initial-value problem of floating bodies without forward speed. Following a treatment by Finkelstein [7] for initial-value water-wave problems, he showed formally how Cummins' decomposition was related to the more general formulation. In the same work, Wehausen also derived the so-called Haskind relations which expressed the exciting forces in terms of the radiation potentials.

The transient motion of a freely floating body was first examined in a consistent manner by

Ursell [16], who considered the problem by superposing wave harmonics. By examining the analytical behavior of the force coefficients in the frequency domain, in particular, near the origin of the complex-frequency plane, Ursell obtained a large-time asymptotic behavior of the response for a semisubmerged circular cylinder. Maskell and Ursell [14] later also computed the entire response for the initial-displacement and initial-velocity problem of this cylinder by using known analytical results of the force coefficient in the frequency domain. Ohmatsu [15] attempted an integral equation formulation based on a source distribution of the unsteady Green function which satisfies the linearized free-surface condition and encountered difficulties caused by eigen frequencies of the interior problem. Adachi and Ohmatsu [2] later explained that the origin of the difficulty is related to the formulation of their integral equation. Daoud [6] considered a closely related problem with an application to study the flow near the bow of a ship in mind. Chapman [4] proposed a novel treatment of such unsteady free-surface problems by representing the fluid motion in terms of a discrete set of wave harmonics. A careful choice of wave-number and frequency distribution must be exercized in order to represent a completely nonreflective exterior fluid domain.

The present work treats the freely floating cylinder problem with a more traditional approach, i.e. via the use of an unsteady Green function that satisfies the linearized free-surface condition. In Section 2, we provide a formulation of the problem based on energy consideration. The integral equation for the velocity potential is derived in Section 3 by using Green's theorem. Particular attention is given to the physical interpretation of various initial-disturbance terms. For a cylinder whose motion starts from rest, we show in Section 4 that the solutions of the initial-displacement and initial-velocity (impulsive-starting) problems are related in a manner very similar to that of a simple harmonic oscillator. Section 5 describes the numerics associated with the solution of this fluid-body interaction problem. Numerical treatment of the unsteady Green function is detailed. Results for the case of circular, triangular, and rectangular cylinders are presented and discussed in Section 6; those for the latter two geometries are heretofore not available in the literature. Also presented are some previously unpublished experimental measurements due to Ito [10] for the case of a semicircular section.

2 Problem formulation

We consider a cylinder of an arbitrary shape floating on the free surface of an ideal fluid. Let Oxy be the coordinate system defined in Figure 1, $\bar{a}$ be the half-breadth of the cylinder, $\bar{\rho}$ the fluid density, and $\bar{g}$ the acceleration of gravity. We shall adopt the convention that all quantities are assumed to be nondimensional unless they are overbarred. The characteristic length, mass, and time used for nondimensionalization will be chosen as $\bar{a}$, $\bar{\rho}\bar{a}^2$, and $(\bar{g}/\bar{a})^{-1/2}$ respectively.

We assume that at time $t = 0$ the cylinder is given either an initial position $y(0)$ with the initial velocity being zero (Problem 1) or an initial velocity $\dot{y}(0)$ with the initial position being zero (Problem 2). The objective is to determine the subsequent vertical motion $y(t)$ of the cylinder. In the context of irrotational flow, it is not difficult to show that the linearized problem for the (nondimensional) velocity potential $\Phi(x, y, z, t)$ is defined by

$$\nabla^2\Phi(x, y, t) = 0 \quad \text{for} \quad (x, y) \text{ in } \mathscr{D}, \tag{1}$$

$$\left.\frac{\partial\Phi}{\partial n}\right|_{\mathscr{B}} = \dot{y}(t)n_y, \tag{2}$$

$$\Phi_{tt}(x, 0, t) + \Phi_y = 0, \tag{3}$$

where $\mathscr{B}$ is the equilibrium position of the body and $\mathbf{n}$ is an outward normal to the fluid. The linearized free-surface elevation $Y(x, t)$ and hydrodynamic pressure $p(x, y, t)$ are related to Φ by:

$$Y(x, t) = -\Phi_t(x, 0, t), \tag{4}$$

$$p(x, y, t) = -\Phi_t(x, y, t). \tag{5}$$

The hydrodynamics problem (1–3) is coupled with the rigid-body dynamics via the boundary condition (2), since $\dot{y}$ itself is unknown and affected by the fluid motion.

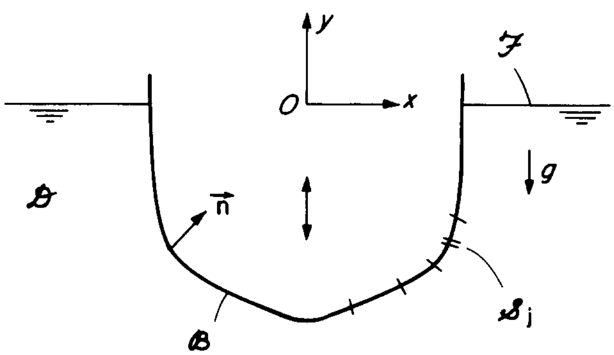


Figure 1. Coordinate system.

Let $E(t)$ be the total energy of the system consisting of the cylinder and the fluid at any time. If E_b and E_f denote the energy of the body and the fluid respectively, then

$$E(t) = E_b + E_f = E(0) \tag{6}$$

or

$$\dot{E}_b = -\dot{E}_f \tag{7}$$

by virtue of the conservative nature of this system. At any given instant, the energy of the body is given by

$$E_b(t) = y^2 + \frac{m}{2}\dot{y}^2, \tag{8}$$

where the first term is potential energy associated with the hydrostatic restoring force and the second the kinetic energy of the body.

The kinetic and potential energy associated with the fluid is given by

$$E_f(t) = \tfrac{1}{2}\iint_{\mathscr{D}} |\nabla\Phi|^2\,dxdy + \tfrac{1}{2}\int_{\mathscr{F}} Y^2(x,t)\,dx, \tag{9}$$

where $\mathscr{F}$ is the part of the x-axis outside of the cylinder. Thus,

$$\begin{aligned} \dot{E}_f(t) &= \iint_{\mathscr{D}} (\nabla\Phi_t \cdot \nabla\Phi)\,dxdy + \int_{\mathscr{F}} \Phi_t\Phi_{tt}\,dx \\ &= \int_{\mathscr{B}} \Phi_t \frac{\partial\Phi}{\partial n}\,ds + \int_{\mathscr{F}} (\Phi_t\Phi_y + \Phi_t\Phi_{tt})\,dx \\ &= \int_{\mathscr{B}} \Phi_t n_y\,ds\,\dot{y}, \end{aligned} \tag{10}$$

where ds represents the differential arc-length element along $\mathscr{B}$. In arriving at the second equality of (10), Gauss' theorem has been used. The third equality is a consequence of (3) and (2).

Differentiating (8) and using (7), we obtain

$$\left(m\ddot{y} + 2y + \int_{\mathscr{B}} \Phi_t n_y\,ds\right)\dot{y} = 0. \tag{11}$$

The quantity in parenthesis could have been derived in a more customary manner using Newton's second law by noting that the force on the body $F(t)$ is the negative of the third term of (11). The present derivation, however, shows more concisely how energy could be transferred back and forth between the body and the fluid.

The complete hydrodynamics problem is therefore the solution of (1–3) subject to the additional differential equation (11).

3. Integral equation for Φ

An integral equation for Φ can be derived using Volterra's method. To accomplish this, it is necessary to introduce an unsteady Green function G of the form:

$$G(P, Q, t-\tau) = \log r + H(P, Q, t-\tau), \tag{12}$$

where r is the distance between the field point $P = (x, y)$ and the source point $Q = (\xi, \eta)$ and τ is a dummy time variable representing the instant G is brought into existence. The function H is to be harmonic in $\mathscr{D}$ and to be so constructed that

$$G_{\tau\tau}(P;\xi,0;t-\tau) + G_\eta = 0, \tag{13a}$$

$$G(P;\xi,0;0) = G_\tau(P;\xi,0;0) = 0. \tag{13b}$$

If we apply Green's second identity to $\Phi_\tau(P, \tau)$ and G, and integrate the resulting expression from $\tau = 0$ to $\tau = t$, then for $P \notin \mathscr{B}$,

$$2\pi[\Phi(P,t) - \Phi(P,0)] = \int_0^t d\tau \int_{\mathscr{B}} ds(Q)[\Phi_\tau(Q)G_\nu(P,Q;t-\tau) - \Phi_{\nu\tau}G] + \int_0^t d\tau \left\{ \int_{-\infty}^{-a} + \int_a^\infty d\xi\, [\Phi_\tau G_\eta - \Phi_{\tau\eta} G]_{\eta=0} \right\}, \tag{14}$$

where $\partial/\partial\nu = \mathbf{n} \cdot (\partial/\partial\xi, \partial/\partial\eta)$. By (3), (4), and (13), the bracketed quantity evaluated at $\eta = 0$ can be written as

$$[\Phi_\tau G_\eta - \Phi_{\tau\eta} G]_{\eta=0} = -\frac{\partial}{\partial\tau}[\Phi_\tau G_\tau + \Phi_\eta G]_{\eta=0} = \frac{\partial}{\partial\tau}[YG_\tau - Y_\tau G],$$

which vanishes when evaluated at the upper limit, $\tau = t$, because of (13b). To avoid solving an integro-differential equation, we integrate the body terms by parts once. Hence,

$$\begin{aligned} 2\pi\Phi(P,t) = {} & 2\pi\Phi(P,0) - \int_{\mathscr{B}} \left[\Phi(Q,0)\frac{\partial}{\partial\nu} - \Phi_\nu\right] G(P,Q,t)\,ds \\ & - \int_{\mathscr{F}} [Y(\xi,0)G_\tau(P;\xi,0;t) - Y_\tau G]\,d\xi \\ & + \int_{\mathscr{B}} \left[\Phi(Q,t)\frac{\partial}{\partial\nu} - \Phi_\nu\right] G(P,Q,0)\,ds - \int_0^t d\tau \int_{\mathscr{B}} \left[\Phi(Q,\tau)\frac{\partial}{\partial\nu} - \Phi_\nu\right] H_\tau\,ds. \end{aligned} \tag{15}$$

An expression related to (15) without the integration-by-parts on $\mathscr{B}$ was first given by Finkelstein [7]. We shall now explain the physical nature of the various inhomogeneous terms in (15). Let

$$\begin{aligned} \mathscr{L}[\Phi] = {} & 2\pi\Phi(P,t) - \int_{\mathscr{B}} \Phi(Q,t)\,G_\nu(P,Q,0)\,ds \\ & + \int_0^t d\tau \int_{\mathscr{B}} \Phi(Q,\tau) H_{\tau\nu}(P,Q,t-\tau)\,ds. \end{aligned} \tag{16}$$

The potential at time t can be decomposed into a sum of three separate effects as follow:

$$\mathscr{L}[\Phi_B] = -\int_{\mathscr{B}} \Phi_\nu(Q,t)\,G(P,Q,0) + \int_0^t d\tau \int_{\mathscr{B}} \Phi_\nu(Q,\tau) H_\tau(P,Q,t-\tau)\,ds, \tag{17a}$$

$$\mathscr{L}[\Phi_{F1}] = -\int_{\mathscr{F}} Y(\xi, 0)\, G_\tau(P; \xi, 0; t)\, d\xi, \tag{17b}$$

$$\mathscr{L}[\Phi_{F2}] = \int_{\mathscr{F}} Y_\tau G(P; \xi, 0; t)\, d\xi + 2\pi\Phi(P, 0)$$

$$-\int_{\mathscr{B}} \left[\Phi(Q, 0)\frac{\partial}{\partial\nu} - \Phi_\nu\right] G(P, Q, t)\, ds$$

$$= -\int_{\mathscr{F}} \Phi(Q, 0)\, G_{\tau\tau}(P; \xi, 0; t)\, d\xi. \tag{17c}$$

The component Φ_B is evidently the fluid motion generated by the body boundary condition for $t \geqslant 0^+$, assuming no previous disturbances existed in the fluid. Equation (17a) is valid even for motion that starts impulsively. Φ_{F1} describes the fluid motion in the presence of the body due to an initial free surface elevation and its subsequent evolution. Φ_{F2} describes the corresponding fluid motion due to an initial velocity on the free surface. The simplification on the right-hand side of (17c) is obtained by using Green's theorem and recognizing that $\Phi(P, 0)$ corresponds to a potential field induced by $Y_\tau(x, 0)$ on $y = 0$. Thus, the combination of Φ_{F1} and Φ_{F2} represents essentially the diffraction potential due to the body during the evolution of a Cauchy-Poisson wave system (cf. Lamb [13]). These points were apparently not clearly stated in the literature.

For the problem at hand, we assume that the fluid was initially at rest. Thus it is only necessary to solve for Φ_B, whose subscript will be omitted hereafter. An integral equation for Φ on $\mathscr{B}$ can be readily obtained by letting P approach $\mathscr{B}$. The net effect is that the factor 2π in (16) is now replaced by π (see e.g., Kellogg [11]). If we define

$$\hat{\mathscr{L}}[\Phi] = \pi\Phi(P, t) - \int_{\mathscr{B}} \Phi(Q, t)\, G_\nu(P, Q, 0)\, ds + \int_0^t d\tau \int_{\mathscr{B}} \Phi(Q, \tau) H_{\tau\nu}(P, Q, t-\tau)\, ds, \tag{18}$$

then the integral equation for Φ on $\mathscr{B}$ is

$$\hat{\mathscr{L}}[\Phi] = -\dot{y}(t) \int_{\mathscr{B}} n_y G(P, Q, 0)\, ds + \int_0^t \dot{y}(\tau)\, d\tau \int_{\mathscr{B}} n_y H_\tau(P, Q, t-\tau)\, ds. \tag{19}$$

Equation (19) must be solved for all t in conjunction with the dynamics equation (11).

The unsteady Green function is available from Finkelstein [7]. If we introduce the complex variables $z = x + iy$ and $\zeta = \xi + i\eta$, with $i = \sqrt{-1}$, we can write

$$G(P, Q, t) = \operatorname{Re}\left[\log(z - \zeta) - \log(z - \bar{\zeta}) - 2\int_0^\infty \frac{dk}{k}\,(1 - \cos\sqrt{k}t)\, e^{-ik(z-\bar{\zeta})}\right], \tag{20}$$

where the overhead bar denotes complex conjugate and Re stands for the real part of the expression. From (20), it is evident that

$$G(P, Q, 0) = \log(r/r_1), \qquad r_1 = \sqrt{(x-\xi)^2 + (y+\eta)^2} \tag{21}$$

which represents the flow about the cylinder and its image above the free surface in an infinite fluid. This is compatible with the first condition in (13b). The 'memory' kernel H_T is thus given by the expression:

$$H_T(P, Q, t-\tau) = \mathrm{Re}\left[2\int_0^\infty \frac{dk}{\sqrt{k}} \sin\sqrt{k}\,(t-\tau)\,e^{-ik(z-\bar{\xi})}\right]. \tag{21}$$

Before discussing the numerical solution of (19), we will show that the solutions of Problems 1 and 2 defined in Section 2 are related to each other in a manner not very different from the case of a simple harmonic oscillator.

4. Relation between initial-displacement and initial-velocity problems

In order to show that the solutions of the initial-displacement problem and initial-velocity problems are related, we first observe the following property of the Volterra operator $\hat{\mathscr{L}}$:

$$\frac{\partial}{\partial t}\hat{\mathscr{L}}[\Phi] = \hat{\mathscr{L}}[\Phi_t] + \int_{\mathscr{B}} ds\,\Phi(Q, 0^+) H_{\nu T}(t-0^+), \tag{22}$$

which can be readily verified by Leibnitz's rule. From (19) we also observe that at $t = 0^+$

$$\hat{\mathscr{L}}[\Phi(P, 0^+)] = -\dot{y}(0^+)\int_{\mathscr{B}} n_y G(P, Q, 0)\,ds. \tag{23}$$

We note that the second term of (22) vanishes if the initial velocity vanishes. Since $G(P, Q, 0)$ defined by (21) represents the infinite-fluid Green function, it is evident from (23) that the nondimensional force $F(t)$ in (11) at $t = 0^+$ is given simply by

$$F(0^+) = -\mu_\infty \ddot{y}(0^+), \tag{24}$$

where μ_∞ is the infinite-fluid nondimensional added-mass coefficient of the submerged portion of the body.

Let $y_1(t)$ be the solution to the *unit* initial-displacement problem and $\Phi_1(P, t)$ be the associated velocity potential, i.e.

$$m\ddot{y}_1 + 2y_1 = -\int_{\mathscr{B}} \Phi_{1t} n_y\, ds, \tag{25}$$

$$\hat{\mathscr{L}}[\Phi_1] = -\dot{y}_1(t)\int_{\mathscr{B}} n_y G(P,Q,0)\,ds + \int_0^t \dot{y}_1(\tau)\,d\tau \int_{\mathscr{B}} n_y H_\tau(P,Q,t-\tau)\,ds, \tag{26}$$

with the initial conditions being

$$y_1(0^+) = 1, \qquad \dot{y}_1(0^+) = 0. \tag{27}$$

If equations (25) and (26) are differentiated once with respect to time, then

$$m\ddot{h} + 2h = -\int_{\mathscr{B}} n_y \phi_t ds, \tag{28}$$

$$\hat{\mathscr{L}}[\phi] = -\dot{h}(t)\int_{\mathscr{B}} n_y G(P,Q,0)\,ds + h(0^+)\int_{\mathscr{B}} n_y H_\tau(t-0^+)\,ds + \int_0^t \dot{h}(\tau)\,d\tau \int_{\mathscr{B}} n_y H_\tau(t-\tau)\,ds, \tag{29}$$

where we have defined $h = \dot{y}_1$, $\phi = \Phi_{1t}$ and made use of (22) on the left-hand side of (29). Now since $h(0^+)$ vanishes because of (27), we conclude immediately that $h(t)$ and $\phi(P, t)$ satisfy the same equations as y_1, but with the initial conditions

$$h(0^+) = 0, \tag{30}$$

$$\dot{h}(0^+) = -2/(m+\mu_\infty), \tag{31}$$

where (31) is the result of (25) evaluated at $t = 0^+$. If y_2 is used to denote the solution to the *unit* initial-velocity problem, then evidently

$$y_2(t) = -\tfrac{1}{2}(m+\mu_\infty)\dot{y}_1(t), \tag{32}$$

which is the relation being sought. The above proof encompasses the special case of a circular cylinder considered by Ursell [16] who noted that $y_2(t) = -\frac{1}{2}\pi\dot{y}_1(t)$ by solving the problem in the frequency domain. Because the mathematical structure of the problem in three dimensions is identical to that of the present, equation (32) is valid for any three-dimensional body also. Thus, the solution for any freely floating body with initial displacement $y(0^+)$ and initial velocity $\dot{y}(0^+)$ can be written as:

$$y(t) = y(0^+)y_1(t) - \tfrac{1}{2}(m+\mu_\infty)\dot{y}(0^+)\dot{y}_1(t). \tag{33}$$

Interestingly, in spite of the complexities of the fluid interaction, (33) is identical to the case of a simple damped harmonic oscillator with the mass replaced simply by the infinite-fluid virtual mass of the section. We note here that the factor $\frac{1}{2}$ in (33) is associated with the inverse of the spring constant.

5. Numerical solution

The numerical solution of the integral equation (19) is sought in conjunction with the dynamics equation (11) at discrete instances of time t_k, $k = 0, 1, 2, \ldots$, with $t_0 = 0^+$. To facilitate the evaluation of the boundary integrals, we employ a polygonal representation of $\mathscr{B}$ by defining a set of grid points (ξ_j, η_j), $j = 1, 2, \ldots, N+1$, which are joined successively by straight-line segments designated here as $\mathscr{S}_j, j = 1, 2, \ldots, N$.

Let $\Phi_i^{(k)}$ be the 'spatial-average' of the potential on $\mathscr{S}_i$ at t_k, and $y^{(k)}$ and $\dot{y}^{(k)}$ be the position and the velocity of the body at the same t. Using the notations introduced, we can now write (19) as

$$\pi\Phi_i^{(k)} - \sum_{j=1}^{N} \Phi_j^{(k)}\alpha_{ij} + \dot{y}^{(k)}\sum_{j=1}^{N} n_j\beta_{ij} = A_i^{(k)}, \qquad i = 1, \ldots, N, \tag{34}$$

where all memory terms have been transposed to the right-hand side and n_j is the value of n_y on $\mathscr{S}_j$.

The coefficients α_{ij}, β_{ij} are integrals associated with the terms $G_\nu(P, Q, 0)$ and $G(P, Q, 0)$ along $\mathscr{S}_j$ respectively. Their treatment is well known (see, e.g. Frank [8]). By definition, the memory integral $A_i^{(k)}$, as observed at the i-th control point, P_i, where (19) is satisfied, is given by the expression

$$A_i^{(k)} = -\int_0^{t_k} d\tau \left\{ \sum_{j=1} \int_{\mathscr{S}_j} \left[\Phi(Q,\tau)\frac{\partial}{\partial\nu} - \dot{y}(\tau)n_y(Q) \right] H_\tau(P_i, Q, t_k - \tau)\, ds(Q) \right\}. \tag{35}$$

In spite of the fact that both Φ and $\dot{y}$ may be relatively slow-varying functions of time, H_τ could oscillate very rapidly, particularly when P_i is close to the free surface and $t_k - \tau$ is large. As an example, we plot in Figure 2 the curly bracketed quantity of (35), designated here as $A^{(k)}(t_k - \tau)$, for the case of prescribed motion: $\dot{y}(t) = \sin \omega t$, with $\omega = 0.5$. For simplicity, the geometry is taken as a semicircle, and N and i are both chosen to be 8. The integrand is shown here for ten successive instances of time during the second cycle of oscillation, i.e. $T \leqslant t_k \leqslant 2T$, with T being the period. The integral $A_i^{(k)}$, which must be evaluated with care, is cross plotted on the right side of the figure, with the vertical axis representing t_k/T. The contribution associated with β_{ij} is also shown. Generally speaking, the term associated with the infinite-frequency kernel is more important than $A_i^{(k)}$ when P_i is near the bottom of the cylinder, and the opposite is true when P_i is near the free surface. This is congruent to the fact that the bottom of the cylinder experiences less wave-motion than the sides.

If the time stepping is chosen so that Φ and $\dot{y}$ do not change rapidly within a given time step, it is possible to integrate out the oscillatory part of H_τ as follows:

$$A_i^{(k)} = -\sum_{m=1}^{k}\sum_{j=1}^{N} [\bar{\Phi}_j^{(m)}\gamma_{ij}^{(m)} - \bar{\dot{y}}^{(m)}n_j\lambda_{ij}^{(m)}], \tag{36}$$

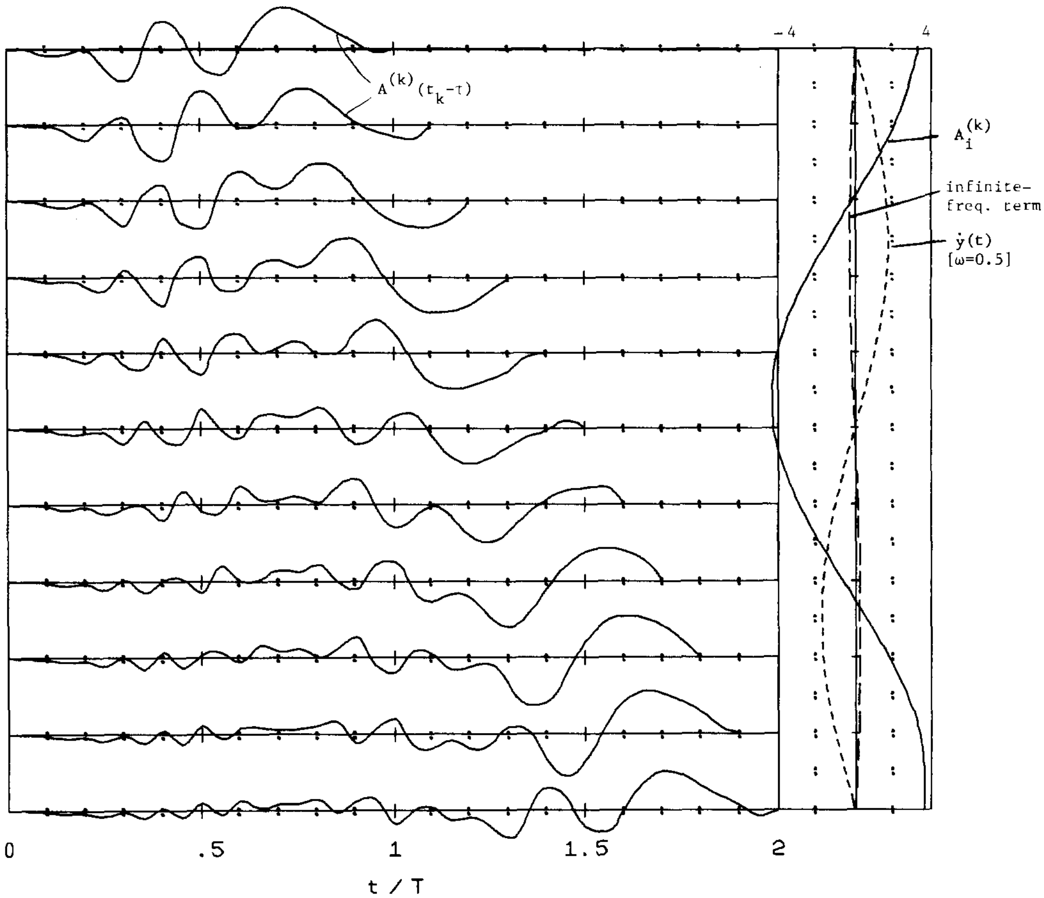


Figure 2. Memory effects due to free-surface at increasing values of t_k.

where

$$\left.\begin{matrix}\gamma_{ij}^{(m)} \\ \\ \lambda_{ij}^{(m)}\end{matrix}\right\} = \int_{t_{m-1}}^{t_m} d\tau \int_{\mathscr{S}_j} dS(Q) \left\{\begin{matrix} H_{\tau\nu}(P_i, Q, t_k - \tau) \\ \\ H_\tau(P_i, Q, t_k - \tau)\end{matrix}\right. \tag{37}$$

A convenient choice of the 'time-averaged' $\bar{\Phi}_j^{(m)}$ and $\bar{\dot{y}}^{(m)}$ is simply $[\Phi_j^{(m-1)} + \Phi_j^{(m)}]/2$ and $[\dot{y}^{(m-1)} + \dot{y}^{(m)}]/2$ respectively. This approximation implies an implicit dependence of $A_i^{(k)}$ on the unknown values of $\Phi_i^{(k)}$ and $\dot{y}^{(k)}$ at t_k; a dependence which, strictly speaking, is absent in (35), since $H_\tau(P_i, Q, 0) = 0$. In practice, it was found that the effects due to these implicit terms are negligible. Nevertheless, we now formally have

$$\pi\Phi_i^{(k)} - \sum_{j=1}^{N} \Phi_i^{(k)}(\alpha_{ij} - \gamma_{ij}^{(k)}/2) + \dot{y}^{(k)} \sum_{j=1}^{N} n_j(\beta_{ij} - \lambda_{ij}^{(k)}/2)$$
$$= -\sum_{m=1}^{k-1} \sum_{j=1}^{N} [\bar{\Phi}_j^{(m)}\gamma_{ij}^{(m)} - \bar{\dot{y}}^{(m)}\lambda_{ij}^{(m)}]$$
$$-\frac{1}{2}\sum_{j=1}^{N} (\Phi_j^{(k-1)}\gamma_{ij}^{(k)} - \dot{y}^{(k-1)}n_j\lambda_{ij}^{(k)}), \quad \text{for} \quad i = 1, 2, \ldots, N, \tag{38}$$

where all known quantities are transposed to the right-hand side. As to be expected, the major computation efforts is in the evaluation γ_{ij} and λ_{ij}. We shall return to that shortly. For the dynamics equation (11), we can rewrite it as two coupled first-order differential equations (involving y and $\dot{y}$):

$$\frac{dy}{dt} = \dot{y}, \qquad \frac{d\dot{y}}{dt} = (-2y + F(t))/m. \tag{39}$$

If a Runge-Kutta third-order algorithm is now applied to (39), we can arrive at the following formulas for advancing the solution:

$$y^{(k)} = Ay^{(k-1)} + B\dot{y}^{(k-1)} + \frac{\Delta t^2}{6m}[F(t_{k-1}) + 2F(t_{k-1/2})], \tag{40}$$

$$\dot{y}^{(k)} + \hat{\Phi}^{(k)}/m = A\dot{y}^{(k-1)} - \frac{2B}{m}y^{(k-1)} - \frac{\Delta t^3}{3m^2}F(t_{k-1}) + \hat{\Phi}^{(k-1)}/m, \tag{41}$$

where

$$A = 1 - \Delta t^2/m, \qquad B = \Delta t(1 - \Delta t^2/3m), \tag{42}$$

$$\hat{\Phi}^{(k)} = \sum_{j=1}^{N} \Phi_j^{(k)} n_j ds_j. \tag{43}$$

In the equations above, Δt is taken as $t_k - t_{k-1}$ which need not be constant. Quantities involving F in (40) and (41) are essentially linear combinations of $\hat{\Phi}$ at two or three time steps when a Lagrange interpolation formula is introduced. Equations (38), (40) and (41) provide $N + 2$ linear equations for the N unknown $\Phi_j^{(k)}, \dot{y}^{(k)}$ and $y^{(k)}$. These are solved simultaneously. By letting $\Delta t \to 0$, we observe that the strongest coupling among these equations is between (38) and (41), where the change in velocity is related directly to the change in $\hat{\Phi}$.

To calculate the integrals of (37), we first exploit the following relations for the integration of any analytic function f in the complex plane:

$$\int_{\mathscr{S}_j} \frac{\partial}{\partial \nu} f(\bar{\zeta})ds(\zeta) = [-i\, f(\zeta)]_{\bar{\zeta}_j}^{\bar{\zeta}_{j+1}}, \tag{44}$$

$$\int_{\mathscr{S}_j} \ell\,(\bar{\zeta})ds(\zeta) \;=\; \frac{\zeta_{j+1}-\zeta_j}{|\zeta_{j+1}-\zeta_j|}\,\tilde{\ell}\,(\zeta)\Big|_{\bar{\zeta}_j}^{\bar{\zeta}_{j+1}}, \tag{45}$$

where $\tilde{\ell}$ is an antiderivative of ℓ. Equations (44) and (45) can be easily derived by using the Cauchy-Riemann relations. If we now introduce the following *indefinite* integrals:

$$J(z-\bar{\zeta},t) \;=\; 2i\int_0^t dt \int_0^\infty \frac{dk}{\sqrt{k}} \sin\sqrt{k}t\, e^{-ik(z-\bar{\zeta})}, \tag{46}$$

$$I(z-\bar{\zeta},t) \;=\; -2i\int_0^t dt \int_0^\infty \frac{dk}{\sqrt{k^3}} \sin\sqrt{k}t\,(1-e^{-ik(z-\bar{\zeta})}), \tag{47}$$

then, by (44) and (45), $\gamma_{ij}^{(m)}$ and $\lambda_{ij}^{(m)}$ can be written in terms of J and I as:

$$\gamma_{ij}^{(m)} \;=\; \mathrm{Re}\,[J(z-\bar{\zeta}_{j+1},t)-J(z-\bar{\zeta}_j,t)]\Big|_{t_k-t_{m-1}}^{t_k-t_m}, \tag{48}$$

$$\lambda_{ij}^{(m)} \;=\; \mathrm{Re}\left\{\frac{\zeta_{j+1}-\zeta_j}{|\zeta_{j+1}-\zeta_j|}\,[I(z-\bar{\zeta}_{j+1},t)-I(z-\bar{\zeta}_j,t)]\Big|_{t_k-t_{m-1}}^{t_k-t_m}\right\}. \tag{49}$$

The integrals I and J as defined by (46) and (47) are related to the Error function of a single argument. To see this, first we note that the *inner* integral of J is a standard integral available from Bateman [3]:

$$\int_0^\infty \frac{dk}{\sqrt{k}} \sin\sqrt{k}t\, e^{-ik(z-\bar{\zeta})} \;=\; \frac{4\Omega}{t}\left[i\frac{\sqrt{\pi}}{2}\,e^{-\Omega^2}\,\mathrm{erf}(-i\Omega)\right]$$

$$\equiv \frac{4\Omega}{t}\,\mathscr{F}(\Omega), \qquad \Omega \equiv \frac{t}{2\sqrt{i(z-\bar{\zeta})}}, \tag{50}$$

where erf is the error function defined in Abramowitz and Stegun [1] and $\mathscr{F}$ is the so-called Dawson integral. We note also that the half plane $\mathrm{Im}(z-\bar{\zeta})\leqslant 0$ is now mapped onto $|\mathrm{Arg}(\Omega)|\leqslant \pi/4$, with the real axis of $(z-\bar{\zeta})$ being the $\pm 45°$ rays of the Ω plane. The function erf oscillates with increasing frequency and amplitudes along $\arg(-i\Omega)=\pi/4$, as $|\Omega|\to\infty$. This behavior, which is reasonably well known in the literature, is primarily responsible for the aforementioned difficulty of integrating $A^{(k)}(t)$ numerically. The time integral of J can be expressed in terms of Ω by noting that $dt/t = d\Omega/\Omega$. Hence,

$$J(z-\bar{\zeta},t) \;=\; J(\Omega) \;=\; 4i\,\mathscr{F}_1(\Omega), \tag{51}$$

where

$$\mathscr{F}_1(\Omega) \equiv 2 \int_0^{\Omega} \mathscr{F}(\Omega)\, d\Omega. \tag{52}$$

If the I integral is integrated with respect to t, followed next by an integration by parts with respect to k, we obtain

$$I = t \int_0^{z-\bar{\zeta}} du \int_0^{\infty} \frac{dk}{\sqrt{k}}\, e^{-iku} \sin \sqrt{k} t - i(z - \bar{\zeta}) J. \tag{53}$$

Using (50) and the fact that $du = it^2 d\Omega/2\Omega^3$, we can reduce (53) to

$$I(z - \bar{\zeta}, t) = 4(z - \bar{\zeta}) \, [2\Omega^2 \mathscr{F}_2(\Omega) + \mathscr{F}_1(\Omega)], \tag{54}$$

where

$$\mathscr{F}_2(\Omega) = \int_{\Omega}^{\infty} \frac{\mathscr{F}(\Omega)}{\Omega^2}\, d\Omega.$$

Thus, the calculation of I and J implies the evaluation of the functions $\mathscr{F}$, $\mathscr{F}_1$, and $\mathscr{F}_2$. All of these functions have the property $\mathscr{F}(\bar{\Omega}) = \overline{\mathscr{F}(\Omega)}$. In actual computations, it is not necessary to evaluate all three quantities simultaneously. In studying some related integrals, Daoud [6] showed that

$$\mathscr{F}_2(\Omega) = \mathscr{F}_1(\Omega) + \frac{\mathscr{F}(\Omega)}{\Omega} - \log \Omega - \tfrac{1}{2} (\gamma + \log 4), \tag{55}$$

where γ is Euler's constant. This relation is particularly useful. For small Ω, it is straightforward to calculate $\mathscr{F}(\Omega)$ and $\mathscr{F}_1(\Omega)$, since both have relatively simple series expansion. For large Ω, an efficient means of evaluating $\mathscr{F}(\Omega)$ is described by Gautschi [9], and a large-Ω expression for $\mathscr{F}_2$ may be derived based on the expression given therein. Thus, $\mathscr{F}_2$ or $\mathscr{F}_1$ follows simply from (55) by using the other two quantities that have been already calculated.

Figures 3a, b and 3c, d illustrate how the functions I and J behave within the 45° wedge of Ω. The perspective plots as shown represent the intersection of the surface defined by the real (or imaginary) part of the functions with vertical planes representing the rays of the Ω plane and with cylindrical surfaces representing the modulus of Ω. As can be seen, these functions, representing the spatial and time integrals of the original function H_T and $H_{T\nu}$, are much less oscillatory and better behaved. As to be expected, the most oscillatory disturbance is still associated with $\arg(\Omega) = \pm\, \pi/4$, corresponding physically to $\text{Im}(z - \bar{\zeta}) = 0$.

Finally, we note that while the coefficients α_{ij} and β_{ij} are not time dependent, the

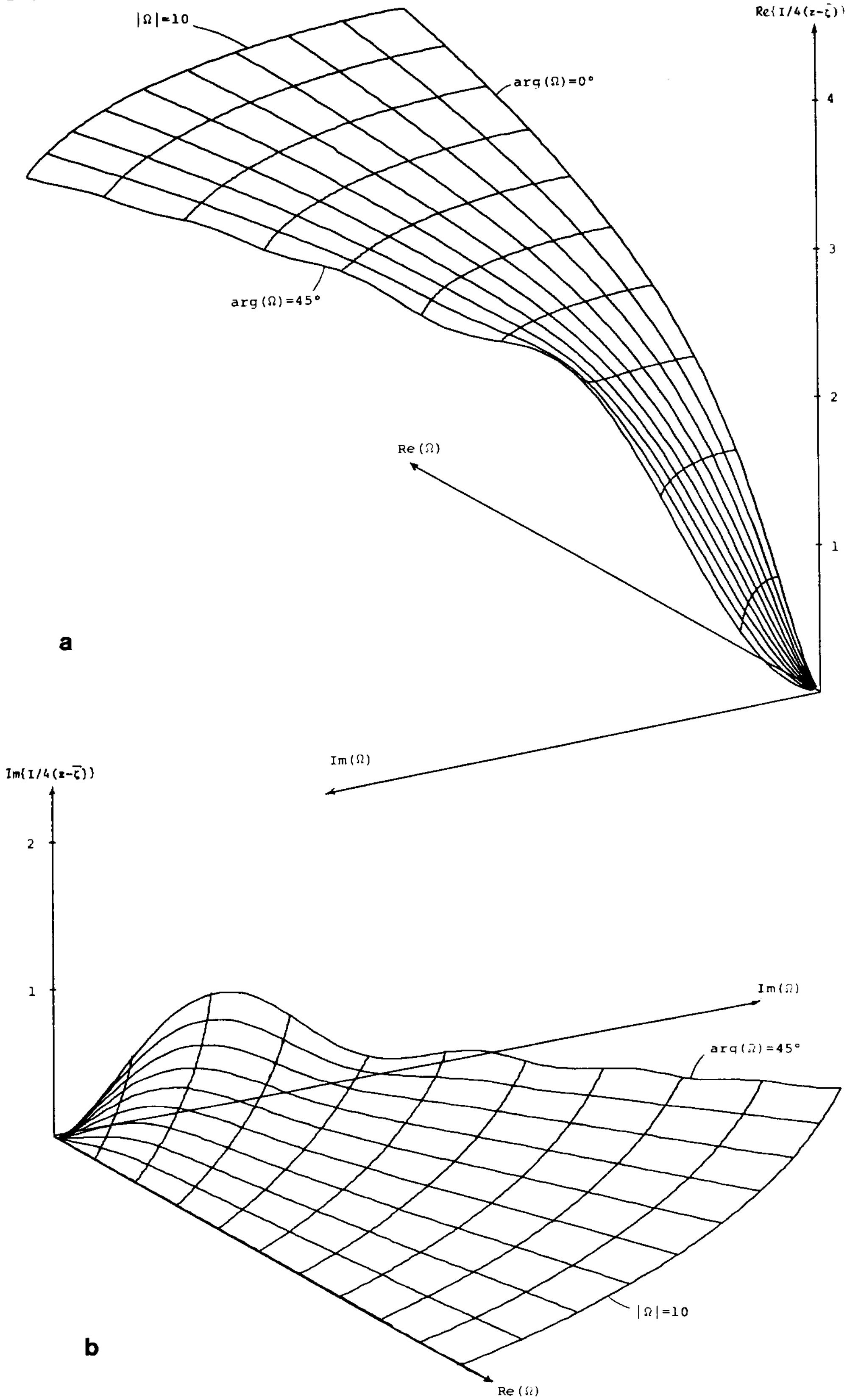
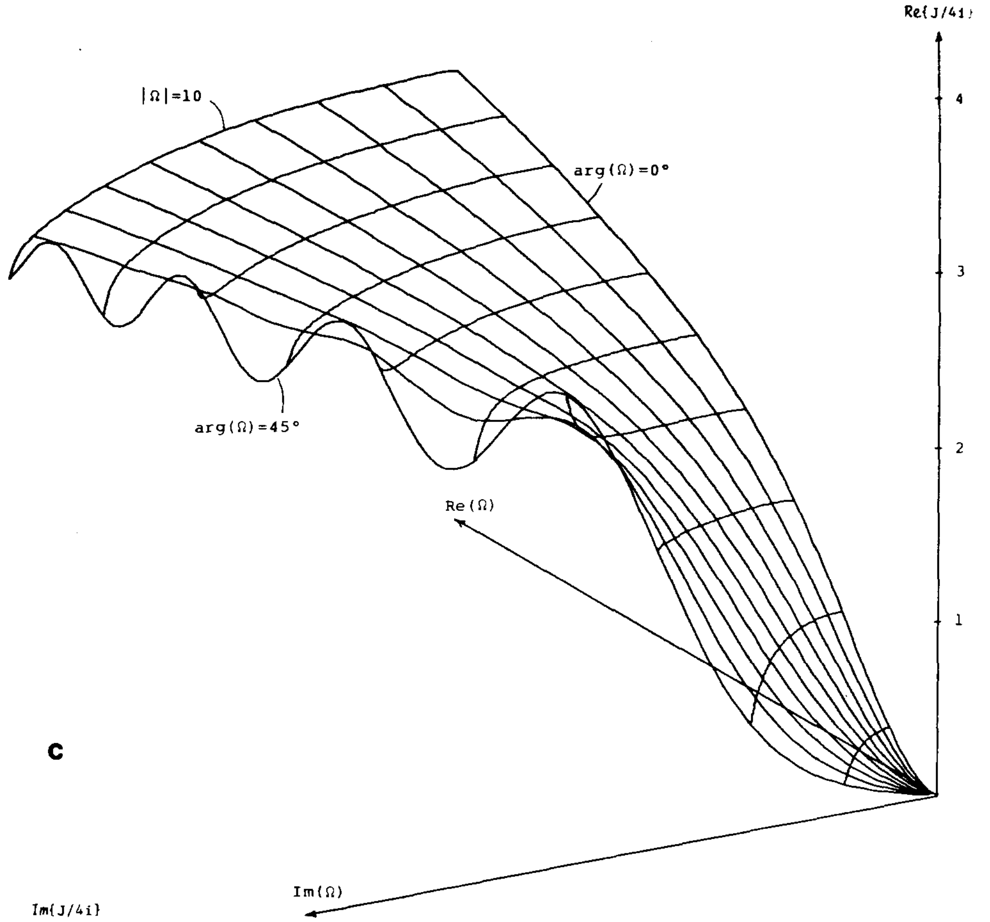


Figure 3a, b. Surface plots of real and imaginary parts of the integral $I(\Omega)/4(z - \bar{\zeta})$.



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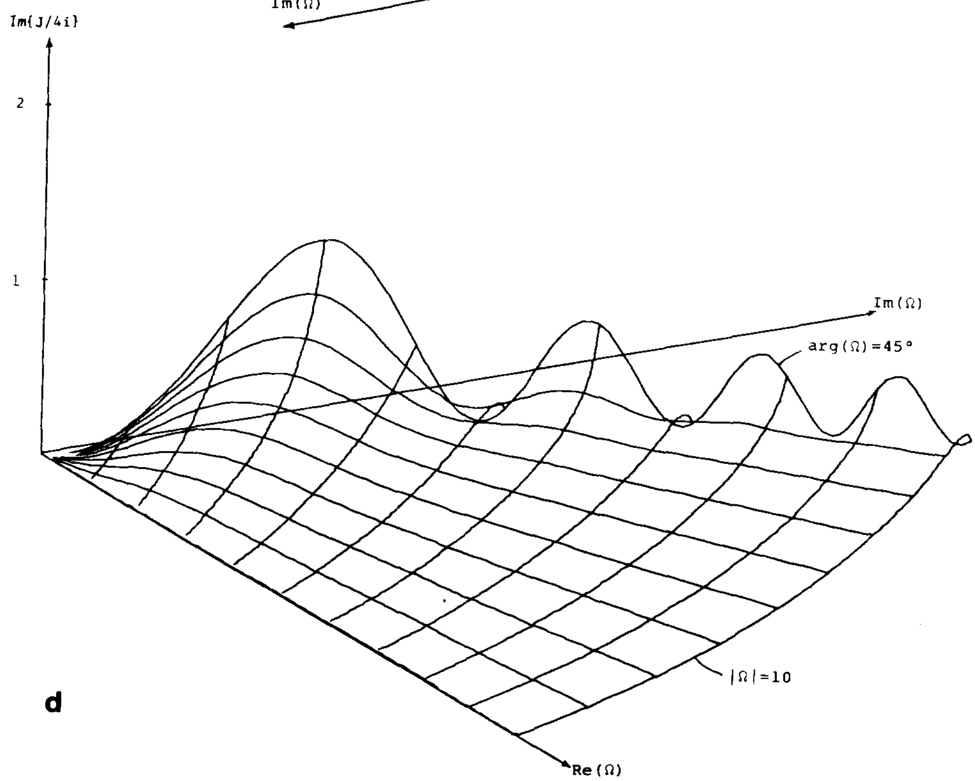

d

Figure 3c, d. Surface plots of real and imaginary parts of the integral $J(\Omega)/4i$.

computation efforts of $A_i^{(k)}$ increases with k like an arithmetic progression. Thus, the calculating of $A_i^{(k)}$, $i = 1, \ldots N$ requires $N^2 k$ evaluations of the quantities γ_{ij} and λ_{ij}; and the total amount of computation effort required to obtain the solution of k steps would be proportional to $N^2 k^2$. However, one extremely useful recurrent property has not yet been pointed out. We note from (37) that $\gamma_{ij}^{(m)}$ and $\lambda_{ij}^{(m)}$ are more properly designated as $\gamma_{ij}^{(k,m)}$ and $\lambda_{ij}^{(k,m)}$, respectively, because of their implicit dependence on t_k. It is easy to deduce that

$$\gamma_{ij}^{(k,m)} = \gamma_{ij}^{(k-1,m-1)}, \qquad \lambda_{ij}^{(k,m)} = \lambda_{ij}^{(k-1,m-1)}, \qquad m \geqslant 2.$$

This effectively reduces the computations by a factor of k. In the actual implementation, the symmetry of the body shape about $x = 0$ was further taken advantage of.

6. Results and discussion

The method of solution described in Section 5 has been applied to obtain the transient response of several types of cylinders. The computer program developed was checked by obtaining the asymptotic forces at large time for prescribed sinusoidal motions of the body and comparing such forces with those based on the frequency-domain solution (see Yeung [18]). Another cross check was made by solving both $y_1(t)$ and $y_2(t)$ separately and verifying that they were related by (32).

Figure 4a shows the response of a circular cylinder with unit initial displacement. Figure 4b shows the manner at which energy is transmitted between the body and the fluid. The corresponding result for the problem with initial velocity is shown in Figures 5a and b. The present results of the displacement are in good agreement with those of Maskell and Ursell [14]. It is worthwhile to note that $E(0^+)$ for the initial-velocity problem is twice that of the initial-displacement problem because of the presence of kinetic energy in the fluid. Such energy is proportional to the infinite-fluid added mass which happens to be identical to the displaced (body) mass. In these figures, the energy transfer rate is seen to oscillate twice as fast as the response, since it is the product of the force and velocity. The maxima on the body-energy curve occur precisely when the body achieves a maximum or minimum position, while the minima correspond to the instances of time that the fluid force acting on the body vanishes.

An experimental assessment of the theoretical solution was carried out by Ito [10] in 1977. Figure 6 is a sketch of the apparatus that was set up across a water tank for recording the transient heave response of a circular tube 6 inches in diameter. The analog signal recorded was digitized and plotted in Figure 7 together with the theoretical results. The theoretical curve slightly over predicts the amplitude of the first and second minimum. Aside from that, the agreement between theory and experiment is excellent.

Theoretical calculations were carried out also for two additional types of shapes: triangular and rectangular sections. Figures 8 and 9 display the response of a 60° and 90°-angle wedge,

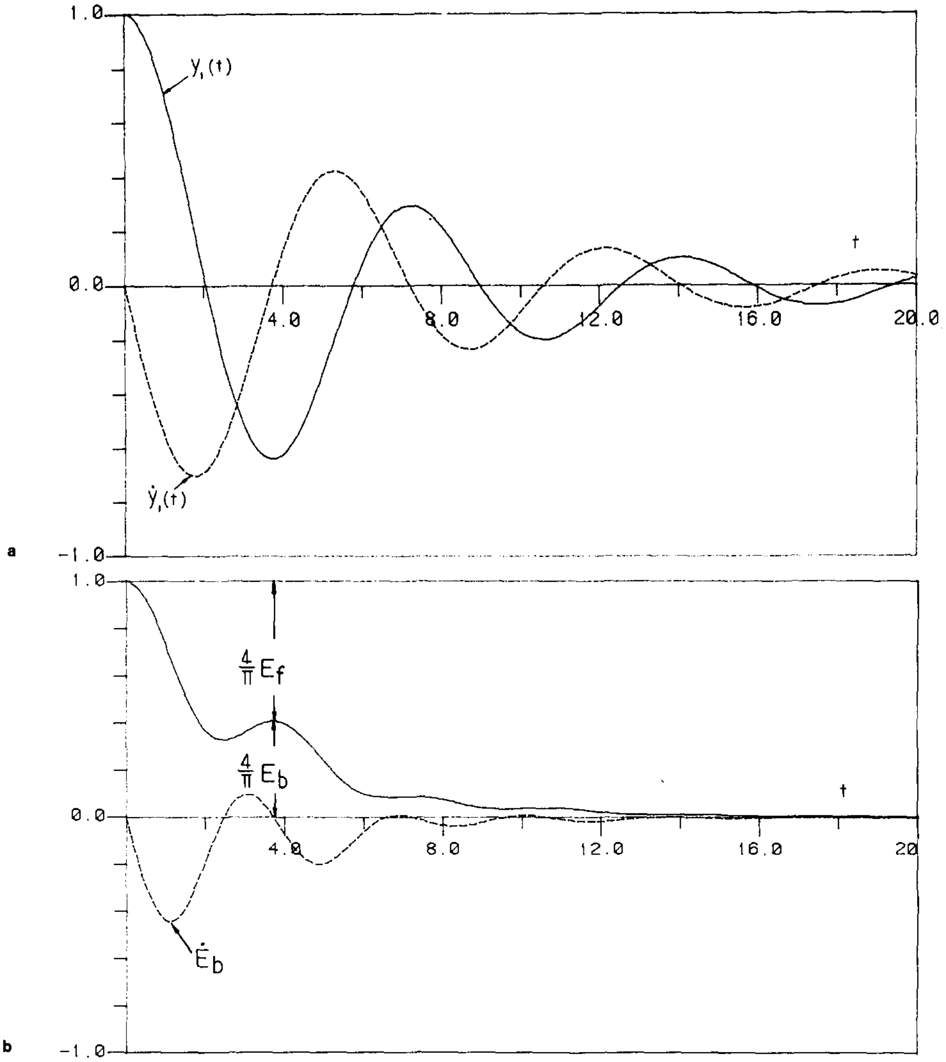

Figure 4. Transient motion and energy transfer rate of a semicircular section due to a unit initial displacement.

respectively. It is of interest to note that the 60° wedge, which has a larger effective mass, oscillates with a longer period than the 90° case. The 90° wedge is however a more effective wave maker by virtue of the fact that body points are on the average closer to the free surface. Its motion damps out very quickly.

Figure 10 shows the initial-displacement response of three rectangular sections with different drafts. The damping characteristics are consistent with what has been pointed out earlier. Aside from that, we observe that the first period of oscillation can be well approximated by one based simply on the infinite-fluid virtual mass of the section.

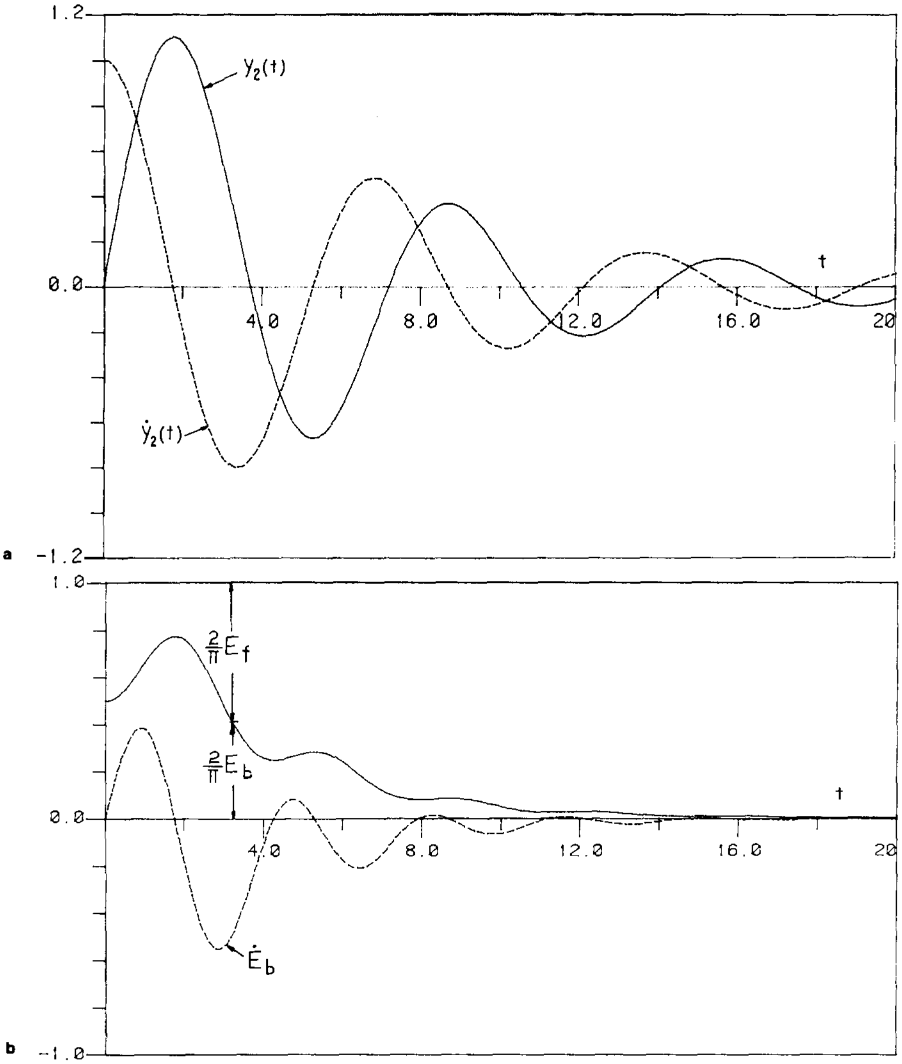

Figure 5. Transient motion and energy transfer rate of a semicircular section due to a unit initial velocity.

Even though the time between successive zero crossings of a typical response curve shown here is not constant, which is an intrinsic property of such transient problems, it is of considerable practical interest to point out that the period between successive maxima or alternate zeros are generally not drastically different from the *damped* 'natural period' of a simple harmonic oscillator of constant coefficients described by the following differential equation:

$$[m + \mu(\omega_d)]\,\ddot{y} + \lambda(\omega_d)\dot{y} + 2y = 0, \tag{56}$$

where ω_d is the solution of the transcendental equation:

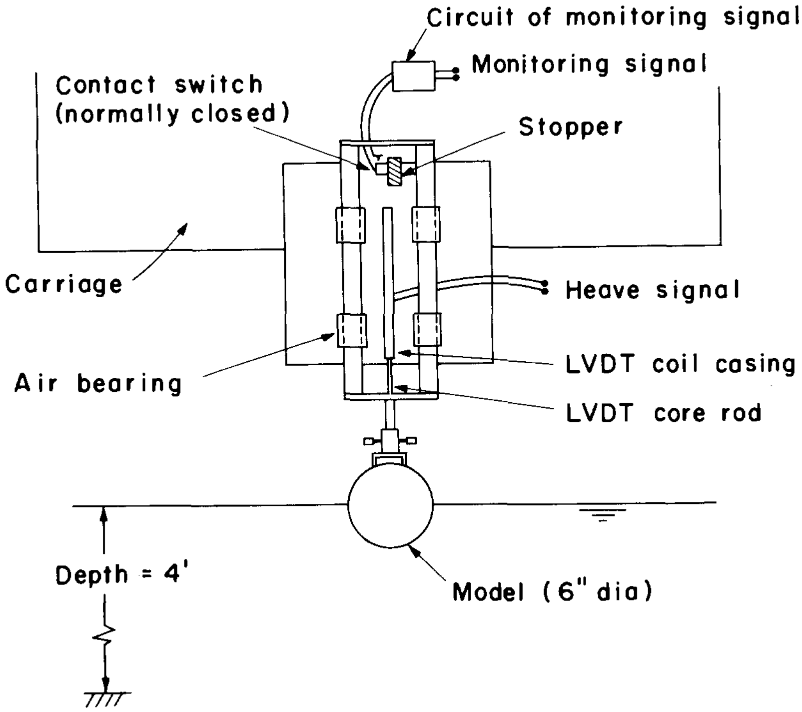

Figure 6. Experimental set-up for measuring transient response.

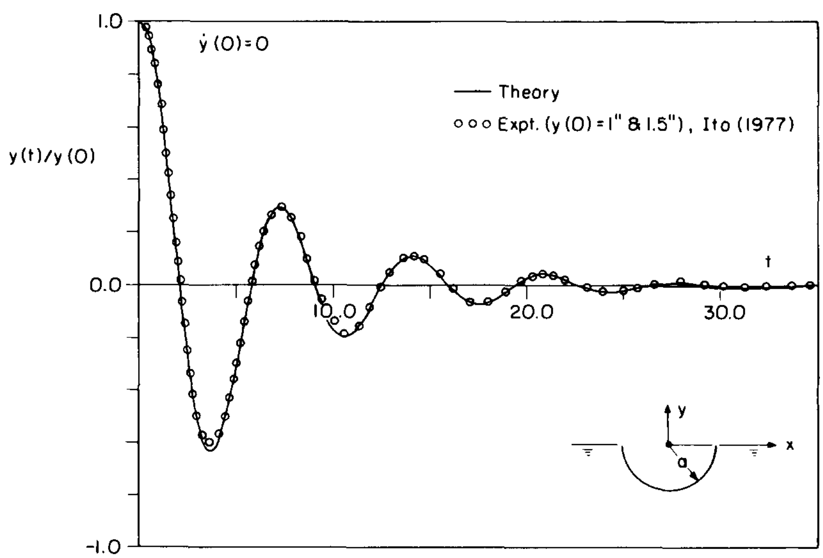

Figure 7. Comparison of experimental and theoretical response for a semicircular section.

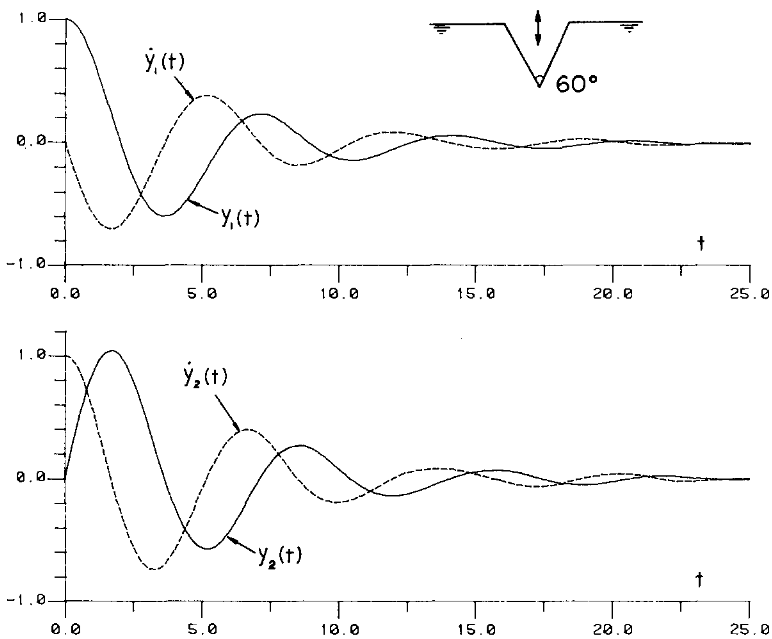

Figure 8. Initial-displacement and initial-velocity responses of a 60 ° triangular wedge.

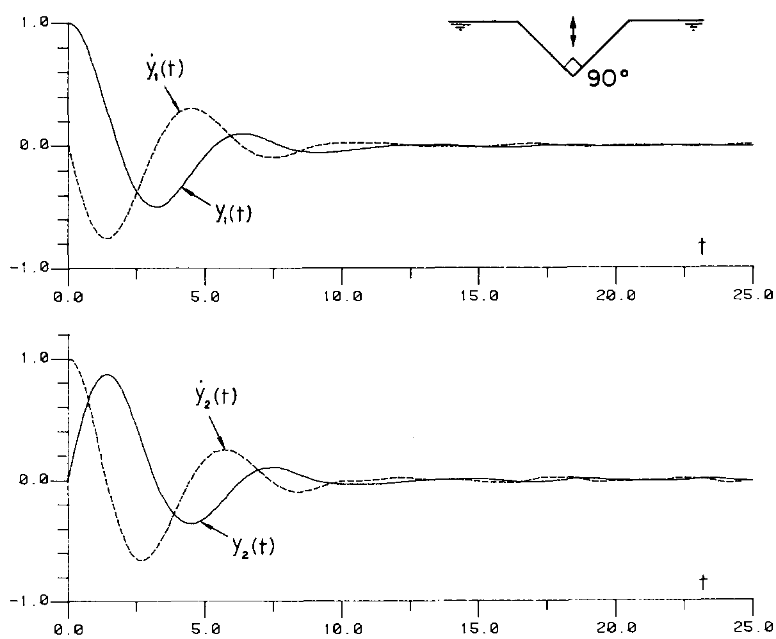

Figure 9. Initial-displacement and initial-velocity responses of a 90 ° triangular wedge.

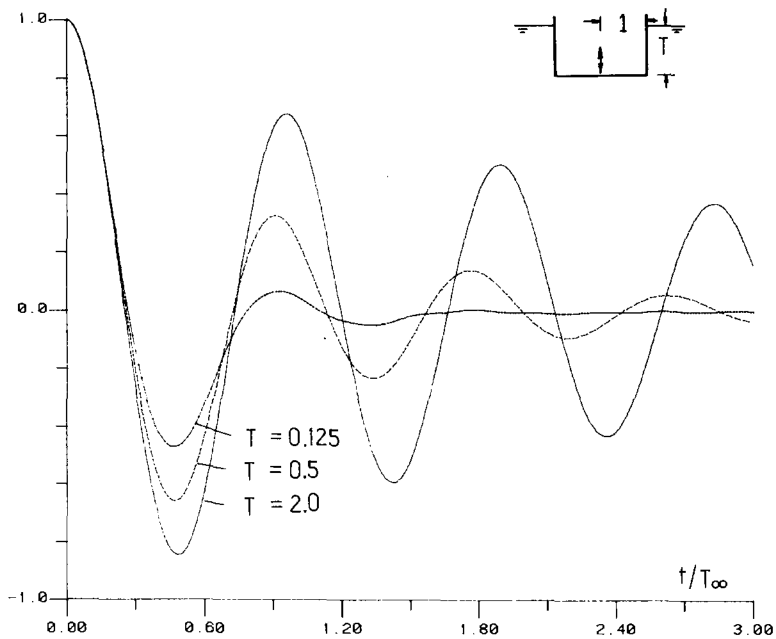

Figure 10. Initial-displacement response of rectangular sections of 3 different drafts ($T_\infty = 2\pi[(m + \mu_\infty)/2]^{1/2}$).

$$\omega_d^2 = \left(\frac{2}{m + \mu(\omega_d)}\right)\left(1 - \frac{\lambda^2}{8(m + \mu)}\right) \tag{57}$$

with μ and λ being the known time-harmonic nondimensional added mass and damping. If the initial-displacement solution of (56) is designated by y_1^*, Figure 11 shows that y_1^* compares well with y_1 for a number of lightly damped shapes. The approximating curves tend to underpredict the first minimum and overpredict the following maximum. Damping coefficients estimated based on the logarithmic decrement of the *exact* transient solution were found to fluctuate about 10% to 20% around the time-harmonic value $\lambda(\omega_d)$. If y_1^* is accepted as a useful approximation, (32) can be used to calculate the initial-velocity response.

The techniques described in this paper can be extended in a relatively straightforward manner to study responses of other modes of motion, with or without the presence of incident waves.

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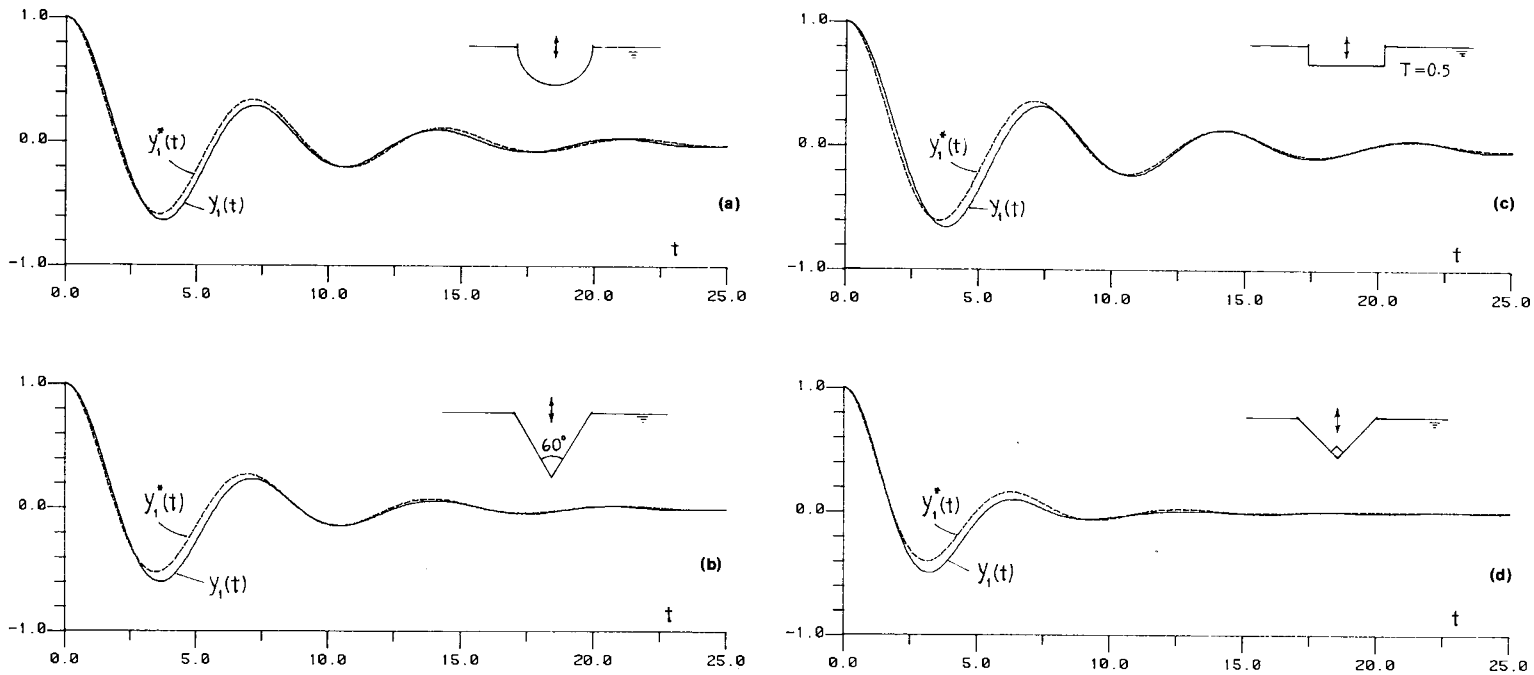

Figure 11. Exact response versus an approximation based on constant hydrodynamic coefficients at 'natural frequency'.

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